High-Speed Computation of Ascent Trajectories within Iterative Loops Using Collocation Methods

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Theme

THE collocation method is a convenient tool for the numerical solution of differential equations.[1] It has found applications in spaceflight trajectory optimization.[2] It is shown in this paper how the method can be adapted to the computation of trajectories within a combined design/performance optimization program for multistage launch vehicles. The parameters of the collocation expansion are adjusted in a single loop, together with the primary control variables of the problem, to satisfy the conditions of a constrained optimum. This results in an order of magnitude reduction in computer time compared with a more traditional approach. Good accuracy is obtained by segmenting the trajectory.

Contents

An approximate solution of the system of differential equations

$$dx/d\tau = f(\tau, x, y) \tag{1}$$

[x n-dimensional state vector, y m-dimensional vector of parameters, $0 \leq \tau \leq 1$, $x(0)$ given] can be found by inserting a general expression

$$x^*(\tau) = \phi[\tau, u, x(0), y] \quad [x^*(0) = x(0)] \tag{2}$$

(u $r \cdot n$-dimensional vector of expansion parameters) into Eq. (1) and matching Eq. (1) at r "collocation points" $\tau_1, \ldots, \tau_r$. This leads to a set of $r \cdot n$ nonlinear equations $\psi = 0$, $\psi^T = (\psi_1{}^T, \ldots, \psi_r{}^T)$ (T = transition to transposed matrix)

$$\psi_i = (\partial\phi/\partial\tau)[\tau_i, u, x(0), y] - f\{\tau_i, \phi[\tau_i, u, x(0), y], y\} \tag{3}$$

They can be solved iteratively with the help of Newton's method. For each iteration one has

$$AU = B$$
$$A = \frac{\partial\psi}{\partial u}, \quad B = -\left[\psi, \frac{\partial\psi}{\partial x(0)}, \frac{\partial\psi}{\partial y}\right] \tag{4}$$

The solution matrix is of dimension $r \cdot n$, $1+n+m$. The first column vector is the linear correction $\delta\tilde{u}$ to u required to satisfy Eq. (3) assuming constant $x(0)$, y. The following two groups of n and m column vectors are the sensitivities $\partial u/\partial x(0)$ and $\partial u/\partial y$, that is the linear corrections required to maintain Eq. (3) when a unit change in some component of either $x(0)$ or y has occurred. A convenient and well-performing expression (2) was found to be

$$x^*(\tau) = x(0) + \sum_{k=1}^{r} u_k \phi_k(\tau)$$
$$\phi_k(\tau) = \tau \;(k = 1) \tag{5}$$
$$= \tau(1-\tau)T_{k-2}(2\tau - 1) \;(k > 1),$$
$$T_k(s) = \cos(k \arccos s), \qquad -1 \leq s \leq +1$$

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in terms of Tchebyshev polynomials. The selection of collocation points is guided by the theory of the method.[1,3,4] A good general-purpose set is given by the zeros of the Tchebyshev polynomial $T_r(2\tau - 1)$ defined by

$$2\tau_i - 1 = \cos\left(\frac{2r - 2i + 1}{2r}\pi\right), \qquad i = 1, \ldots, r \tag{6}$$

In an optimization problem let $\psi^{(0)}$ denote the vector of engineering constraints (including payoff) and $u^{(0)}$ the vector of control variables. Adjustments of the controls is often based on a relationship

$$\delta\psi^{(0)} = [\partial\psi^{(0)}/\partial u^{(0)}]\,\delta u^{(0)} \tag{7}$$

using linear derivatives. The collocation method could be simply applied by adjoining the collocation expansion coefficients u and the collocation equations $\psi = 0$ to the vectors $u^{(0)}$, $\psi^{(0)}$. The following approach is more efficient and allows for segmentation of the trajectory. Consider a sequence of trajectory segments i with initial state vector $x^{(i)}(0)$, collocation expansion parameters $u^{(i)}$ and final state vector $x^{(i)}(1)$. Suppose $\delta\tilde{x}^{(i)}(0)$,

$$R^{(i)}(0) = \frac{\partial x^{(i)}(0)}{\partial y}$$

are given indicating the variation of $x^{(i)}(0)$ to be expected because of violations of the collocation equations in segments $j < i$, and the sensitivities of $x^{(i)}(0)$ relative to trajectory parameters y [which are in turn functions of the primary control variables $u^{(0)}$]. Solving Eq. (4) for the current segment yields $\delta\tilde{\tilde{u}}^{(i)}$, $[\partial u^{(i)}/\partial x^{(i)}(0)]$, $[\partial u^{(i)}/\partial y]$. From this information one computes and stores

$$\delta\tilde{u}^{(i)} = \delta\tilde{\tilde{u}}^{(i)} + \frac{\partial u^{(i)}}{\partial x^{(i)}(0)}\delta\tilde{x}^{(i)}(0)$$
$$V^{(i)} = \frac{\partial u^{(i)}}{\partial y} + \frac{\partial u^{(i)}}{\partial x^{(i)}(0)} R^{(i)}(0) \tag{8}$$

so that, on the next iteration, one can update

$$u^{(i)} \leftarrow u^{(i)} + \delta\tilde{u}^{(i)} + V^{(i)}[\partial y/\partial u^{(0)}]\,\delta u^{(0)} \tag{9}$$

The starting values for the next trajectory segment of the current trajectory are obtained from

$$\delta\tilde{x}^{(i+1)}(0) = \delta\tilde{x}^{(i)}(1) = \delta\tilde{x}^{(i)}(0) + \frac{\partial x^{(i)}(1)}{\partial u^{(i)}}\delta\tilde{u}^{(i)}$$
$$R^{(i+1)}(0) = R^{(i)}(1) = R^{(i)}(0) + \frac{\partial x^{(i)}(1)}{\partial u^{(i)}} V^{(i)} \tag{10}$$

The sequence is started with $\delta\tilde{x}^{(1)}(0) = 0$, $R^{(1)}(0) = 0$ for segment 1. If a constraint $\psi^{(0)}$ is defined in terms of $x^{(i)}(1)$ and $u^{(0)}$, one has the relationship

$$\delta\psi^{(0)} = \left(\frac{\partial\psi^{(0)}}{\partial u^{(0)}} + \frac{\partial\psi^{(0)}}{\partial x^{(i)}(1)} R^{(i)}(1)\frac{\partial y}{\partial u^{(0)}}\right)\delta u^{(0)} \tag{11}$$

replacing Eq. (7).

This method was mechanized for use in a ballistic missile design/performance optimization program. The following numerical results are based on the special case of trajectory optimization for a fixed vehicle design and demonstrate the essential features of the approach. The basic functions (5) and the Tchebyshev points (6) were used. A spherical nonrotating

Table 1 **Trajectory optimization examples**

No. of points per segment	Optimum impact range (nm) (error)	CPU Time per full trajectory including derivatives (Sec. UNIVAC 1108)
2, 2, 2	+105.2	0.9
4, 4, 4	−63.2	1.1
8, 4, 2	−1.0	1.2
10, 6, 4	+0.9	1.6
10, 10, 10	+1.8	2.2
Traditional (Runge-Kutta + Adjoint Equations)	0.0	~15.0

Earth was assumed. The equations of motion for each stage are (velocity v, flight-path angle γ, altitude h, longitude λ)

$$\begin{aligned}\dot{v} &= -g\sin\gamma + g_0\left[\frac{T_v - p\,Ae}{W}\cos\alpha - \frac{D}{W}\right]\\ \dot{\gamma} &= \left(\frac{v}{r} - \frac{g}{v}\right)\cos\gamma + \frac{g_0}{v}\frac{T_v - p\,Ae}{W}\sin\alpha\\ \dot{h} &= v\sin\gamma\\ \dot{\lambda} &= (v/r)\cos\gamma\end{aligned} \tag{12}$$

with the auxiliary equations

$$\begin{aligned}W &= \text{function of } I_{sp}, \int_0^\tau T_v\,d\tau\\ r &= R_E + h\\ g &= g_0(R_E{}^2/r^2)\\ D &= \tfrac{1}{2}\rho v^2 C_D A\\ \alpha &= \text{function of } y, \gamma, \lambda\end{aligned} \tag{13}$$

The air density ρ, static pressure P, and speed of sound a_L are defined as functions of h (1962 Standard Atmosphere). Vacuum thrust is defined as a continuous, piecewise linear function of normalized stage time $\tau = t/t_B$, burn time t_B. C_D is a continuous piecewise linear function of Mach number $M = v/a_L$.

The function of the angle-of-attack is determined by the steering laws used. The components of y are the trajectory steering parameters for all stages. Each stage is defined by (average) specific impulse I_{sp}, vacuum thrust function T_v, exit area Ae, burn time t_B and initial weight W_0. A is the aerodynamic reference and g_0, R_E are the sea-level gravity and the radius of the Earth, respectively.

The program was used to maximize the impact range of a hypothetical three-stage design by varying trajectory steering parameters subject to constraints on the angle of attack and dynamic pressure at Stage 1 burnout, and a constraint on re-entry angle. The projected gradient method was used for the optimization.

First results showed surprisingly large errors in the state variables, especially velocity. This was traced back to the Stage 1 thrust profile which typically displays a very steep tailoff region towards burn-out resulting in a sudden drop in the velocity time-derivative. The transformation

$$v_L = v - v_I \tag{14}$$

resulting in the use of total velocity loss v_L as a state variable instead of velocity itself proved to be a very effective remedy. v_I is the ideal velocity

$$v_I = g_0 I_{sp} \log\frac{W(0)}{W(\tau)} \tag{15}$$

and can be exactly determined at each collocation point by integrating thrust. Some representative results are shown in Table 1.

Very smooth convergence from poor initial nominals was obtained in all cases. The computational effort increases with an increasing number of collocation points, as the required algebraic manipulations involve larger arrays and matrices. Typical reduction in running time as compared with the traditional approach is in the order of one magnitude. Even the crude result obtained by evaluating the differential equations at only two points per segment is already an excellent approximation to the true optimum. The accuracy increases with an increase in the number of points, especially for the critical Stage 1. However, the rate of approaching the true optimum by adding more points becomes relatively small from a certain number of points on.

The accuracy of the trajectory representation is about 2 nm impact range error for 10 collocation points in Stage 1. The convergence in accuracy with a further increase in the number of collocation points is very slow in velocity. This was found to be due to a steep increase of the drag coefficient across the Mach 1 region.

References

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